IDENTIFICATION OF WINNING STRATEGIES USING QUANTUM GAME THEORY: AN APPLICATION TO INDIAN FIELD AND BOARD GAMES

A synopsis

Submitted in the partial fulfilment of the requirement for the dissertation in

Doctor of philosophy

In

Physics

UNDER THE SUPERVISION OF:

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FEBRUARY - 2019

INTRODUCTION

Game theory is a branch of mathematics concerned with the analysis of strategies for dealing with competitive situations where the outcome of a participant's choice of action depends critically on the actions of other participants. Classical game theory finds application in economy, psychology, ecology, and biology [32, 33]. According to game theory, a game consists of a set of players, a set of rules which dictate what actions a player can take, and a payoff function specifying the reward for a given set of played strategies. In other words, it is a triple $< n, \Omega, p >$ where n is the number of players, $\Omega = \times_k \Omega_k$ with $1 \leq k \leq n$ such that each set Ω_k is the set of strategies for the k^{th} player, and $p : \Omega \rightarrow r^n$ such that each $p_k(.)$ With $1 \leq k \leq n$ is the payoff function for the k^{th} player. Without loss of generality, we can imagine the existence of a referee who computes the corresponding payoff function after he receives the strategies being played by each of the players. This formal structure includes all classical games and all quantum games. In other words any game, whether classical or quantum, is fully described by the corresponding triple $< n, \Omega, p >$. On the other hand, given any triple $< n, \Omega, p >$, it is not hard to imagine a purely classical game that might be associated to it. [41]

Game theory, originally developed in the 1940s [42] and 1950s [43–45], has since been the source of important contributions in fields such as economics [46], political science [47], biology [48], and computer science [49]. The addition of quantum information concepts into games led to the formalization of quantum games [50, 51]. Since their introduction, quantum games have been studied in a variety of contexts. With the growing prevalence of quantum computers and quantum networks, quantum games emerge as strong candidates for real world applications in quantum security protocols [52], distributed quantum computing algorithms [53], or improving the efficiency of classical network routing algorithms [54].

In most of the cases quantum description of the system provides advantages over the classical situation. [55] for example, Simon's quantum algorithm for identifying the period of a function chosen by an oracle is more efficient than any deterministic or probabilistic algorithm [56](Simon, 1994); Shor's polynomial time quantum algorithm for factoring [57](Shor, 1994) and the quantum protocols for key

distribution devised by wiener, Bennett and brassard, and Ekert are qualitatively more secure against eavesdropping than any classical cryptographic system [58, 59](Bennett and Brassard, 1984; Ekert, 1991).

Game theory, the study of (rational) decision making in conflict situation, seems to ask for a quantum version. For example, games against nature [60] (Milnor, 1954) should include those for which nature is quantum mechanical. Does quantum theory present more subtle ways of playing games? Classical strategies can be pure or mixed: why cannot they be entangled? Can quantum strategies be more successful than classical ones? And if the answer is yes, are they of any practical value? It is quite intriguing that von Neumann is one of the founders of both modern game theory [61] (von Neumann and Morgenstern, 1953) and quantum theory. Is that a meaningful coincidence?

The development of quantum technology has led to some pertinent questions, like the following [62]: what will quantum information technology bring to daily life in the future? What will happen when quantum networks and quantum entanglements are not only available for scientists, but also accessible to the general public? What could be the conditions on which the quantization of classical games can bring full cooperation even when the information of payoff is incomplete?

Quantum versions [64-66] of existing games [63] offer additional strategies to the players—and resolve dilemmas that occur in the classical versions. As it is possible to recast any algorithm (classical or quantum) as a game characterized by strategies and rules, it is reasonable to believe that the quantum mechanical formulation of existing games can also be helpful in gathering a deeper understanding of quantum algorithms and quantum information processing. It has even been argued that performing of experiments in physics can be viewed as simply playing a ‘game’ against nature in which the observer tries to maximize the information obtained from the system under consideration. Eventually, such studies may even shed light on the great divide between classical and quantum physics [67].

So in what sense are quantum games any ‘better’ than classical games [41]? We believe that the answer lies in the issue of efficiency. Although any game could be played classically, the physically feasible ones form a very restricted subset. As we have seen, playing a game is ultimately about information exchange between the players and the referee - hence if a particular game requires you to submit an infinite

amount of information before the payoff functions can be computed, it will not be a playable game. In other words, we are interested in those games which require only a finite amount of resources and time to play. Hence a connection can be made between this consideration and the study of algorithms. In the study of computation, we learn that there are computable functions which may however not be computed efficiently. Shor's great contribution to information theory was to advance the boundary of the set of efficiently computable functions [68]. This naturally begs the question as to whether it is more efficient to play games quantum mechanically than classically.

LITERATURE REVIEW

The insight gained from the literature reviewed is organised and presented under the following sub-heads.

GAME THEORY

Game theory was formulated by von Neumann to understand the competitive and cooperative behaviour of two competing parties which hold particular interests [6]. Game theory is not just about finding winning strategies for gambling games. We can apply it to many socially relevant strategic situations in day to day life like social sciences, sports, environmental sciences, biological sciences and many more to find an optimal solution. Game theory has many points where Nash equilibrium and Pareto optimality either cease to exist at all or do not exist together.

Game theory has varying applications ranging from the theory of financial markets, construction of quantum computers, quantum communication, quantum cryptography, data transfer, and theory of auctions and decoherence [3]. Some mathematical problems can be rephrased as games. The problem of searching for a particular entry in a database, for example, can be turned into a game in which players compete to see who finds the entry first. Algorithms for solving the problem then become strategies for playing the game. Finding a quantum algorithm that solves the problem faster than its classical counterpart is equivalent to identifying a quantum strategy that can beat the classical alternative. With the recent interest in quantum computing and quantum information theory, there has been an effort to recast classical game theory using quantum probability amplitudes, and hence study the effect of quantum superposition, interference and

entanglement on the agents' optimal strategies. Apart from unsolved problems in quantum information theory, quantum game theory may be useful in studying quantum communication since that can be considered as a game where the objective is to maximize effective communication. A general theory of quantum games has the potential to be useful in many situations that arise in quantum cryptography, computational complexity, communication complexity, and distributed computation.

QUANTUM GAME THEORY

In quantum game theory (QGT) Nash equilibrium and Pareto optimality can exist together. A famous example is prisoners' dilemma. In classical prisoners' dilemma, there is no single strategy that obeys both Nash equilibrium and Pareto optimality. We will discuss quantum version of game theory. The motivation for it arises from the fact that game theory has pure strategies as well as mixed strategies which are probabilistic mixture of pure strategies. What if we could use quantum mechanics to make novel strategies using, say entanglement and superposition? Would this enlarged set of strategies be more efficient than classical strategies? And if yes, what could be the possible applications of quantum game theory? Might be in cryptography and communication [7]. Weisner was the first to coin the term quantum money [2], Vaidman used the term game in quantum context [3], and Meyer [4] and Eisert et al [5] first formulated their problems in game theory formalism. And now a wide range of connections are established between quantum games and other fields e.g., bell non-locality [8] and quantum logic [9] quantum game theory claims to offer a solution to famous prisoners' dilemma which obeys both Nash equilibrium and Pareto optimality. [10]

Quantum game theory presupposes [11]:

- Superposed initial states,
- Quantum entanglement of initial states,
- Superposition of strategies to be used on the initial states.

APPLICATIONS OF QUANTUM GAME THEORY

Quantum games may find many interesting applications. Classical game theory has application in economy, psychology, ecology, and biology [32, 33].also it is based on probability to a large extent so there

are fair chances to generalize it to the domain of quantum probabilities. Many problems of quantum information theory can be formulated as a game, like eavesdropping in quantum channel communication [34-36] and optimal cloning [37]. Meyer [8] demonstrated that quantum mechanics can be used for winning some zero-sum unfair games.

Eisert, Wilkens and Lewenstein [13] and Marinatto, weber [14] proposed a prescription for quantization of games. It can be applied to any 2 x n game (each player has n strategies). This prescription has been used for quantizing various classical games like prisoner's dilemma [13, 15], the Monty hall problem [16, 17], battle of sexes [18, 19, 20], stag hunt game [21], rock, scissors and paper [22, 23], coordination problem [24], duopoly problem [25] and duels [26]. Lambertini claims that quantum mechanics and mathematical economics are isomorphic [27]. Therefore one should expect that various quantum tools such as quantum morphogenesis [28] would be invented and used to describe social phenomena. Arfi proposed to use quantum game for wide spectrum of problems in political sciences [29].

DESIGNING A QUNTUM GAME

Incidentally, john von Neumann is one of the founders of both game theory [12] and quantum theory. While designing quantum games the following five questions need due consideration [8]

- Is the idea of quantum games feasible?
- Under what conditions some players may be able to take advantage of quantum phenomena?
- Are there genuine quantum games that have no classical counterparts or origin?
- Are protocols for playing quantum game against human player secure against cheating?
- Can the formalism be generalized to include other non-Boolean logic based systems?

Many cryptographic and information processing problems can be reformulated in game-like setting. Quantization of a standard classical game would need defining quantum strategies for the game. There is a belief that if quantum games can be applied outside the laboratory, solution for many of our technical problems could be found. Hence we can say that quantum games might prove out to be a very important tool for analysing complex systems. In future we might be able to play trillions of games simultaneously using massive parallel DNA processing.

PRISONER'S DILEMMA [69]

Prisoner's dilemma is an oft quoted example of classical game theory. Typically, in a two-player game between Alice and Bob, where each has two moves available, denoted by *c* (cooperate or confess) and *d* (defect or deny), the payout matrix is represented as under [79]:

Alice/bob	C	D
C	(3,3)	(0,5)
D	(5,0)	(1,1)

The dilemma is that (*d*, *d*) is the dominant equilibrium (neither player can profit from a *unilateral* change), but both players would prefer (*c*, *c*).

QUANTIZATION PROCESS:

At quantum level, the actual quantization occurs when Alice and Bob indicate their choices by performing a certain operation on a qubit. The two qubits given to them are in an entangled state (otherwise, the quantum version is no different from the classical game). Alice and bob still have choices *c* and *d* corresponding to the classical decisions to confess or deny. However, there are more choices available to them. In particular, there is a move, call it *q*, which is in essence a superposition of confessing and denying. The payout matrix including *q* can be constructed from eq. (17) of [70].,

Alice/bob	C	D	Q
C	(3,3)	(0,5)	(1,1)
D	(5,0)	(1,1)	(0,5)
Q	(1,1)	(5,0)	(3,3)

The entries where both Alice and Bob choose *c* or *d* remain the same, so that the classical version is represented as a sub-game. However, the solution (*d*, *d*) is clearly no longer an equilibrium point since each player can do better by *unilaterally* switching to *q*. In fact, (*q*, *q*) is the solution of the new game defined by the payout matrix (2).

Thus, if one party applies *q* and the other uses one of the two classical choices *c* or *d* then the payout is as if the other player had made the other classical choice *d* or *c*. The other player's classical choice can be changed by choosing the move *q*. If both players choose *q,* then the payout is as if both players had

confessed.

THE COIN FLIPPING GAME [7]

In the coin flipping game, typically, there are two players p and q. Q places a coin in the box heads up. They will take turns one after the other successively. Players can choose to flip or not to flip. At the end of the game if the coin is heads up, q wins, else p wins.

This is a two-person zero-sum strategic game which might be analysed using the payoff matrix [31]:

	NN	NF	FN	FF
N	(-1,1)	(1,-1)	(1,-1)	(-1,1)
F	(1,-1)	(-1,1)	(-1,1)	(1,-1)

Where the rows and columns are labelled by p's and q's pure strategies (moves), respectively; f denotes a flip and n denotes no flip; and the numbers in the matrix are Picard's payoffs: 1 indicating a win and -1 a loss of a one currency unit. Q's payoffs can be obtained by reversing the signs in the above matrix.

There is no deterministic winning strategy. Best (mixed strategy) would be to flip with probability ½, don't flip with probability ½. Expected payoff is 0. General (probabilistic) strategy is to flip with probability p, don't flip with probability 1-p.

QUANTUM COIN FLIP

In quantum coin flip, there would be two players p and q. Instead of using a coin there would be an electron with two spin values, u (up) and d (down). P would set up the electron in the spin up state. They would be given turns; q, then p and then q. At the end of the game, if electron is still in the spin up state, q would be the winner [80].

An example of strategy that could be applied is on the first turn, q flips the spin, then p does not flip and q does not flip. The resultant state of spin is u, d, d, d, so p wins.

The coin is represented by a qubit, where |0> represents heads up, and |1> represents tails up. Initial state is given by

$[\rho]=|0><0|=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

$$\rho_0 = |0\rangle\langle 0| = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

A two dimensional vector space v can be defined with basis (u, d). Players' strategies are represented by 2x2 matrices.

Matrix for flipping of the coin is given by

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Matrix for not flipping of the coin is given by

$$N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mixed winning strategy is given by a linear combination of f and n

$$pF + (1-p)N = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$$

Player flips the spin with probability p ϵ [0, 1]

Q's winning strategy is given by

$$U_1 = U_3 = U(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = H$$

After q's first move:

$$\rho_1 = U_1 \rho_0 U_1^\dagger = \tfrac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

After p's move:

$$\rho_2 = pF\rho_1 F^\dagger + (1-p)N\rho_1 N^\dagger = \tfrac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

After q's second move:

$$\rho_3 = U_3 \rho_2 U_3^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The (mixed) quantum equilibrium:

$$([pF + (1-p)N], [U(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}), U(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}})])$$

With exp. Payoff value between 1 and q.

If p uses classical strategies and q utilizes quantum strategies, then it can be proved that q will always win the game no matter which classical strategies p applies.

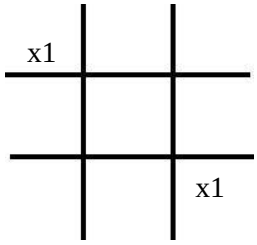
In an otherwise fair zero-sum coin toss game, a quantum player can always win against his /her classical opponent if he/she adopts quantum strategies [38, 40]

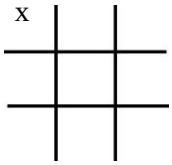
QUANTUM TIC TAC TOE [1]

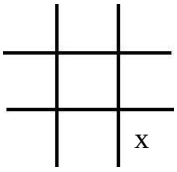
Quantum tic tac toe is the quantum version of the famous classical tic tac toe. In quantum version the moves are superposed.

SUPERPOSITION:-

Superposition is the state of an object to be at more than one place at the same time, which on measurement collapses to a single state.

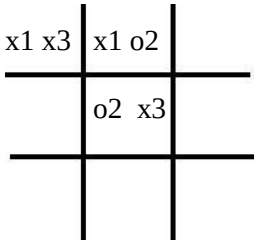


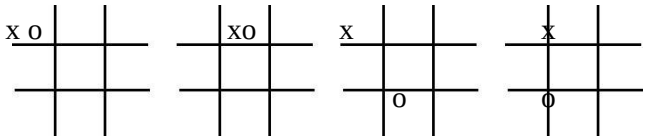




Every single move corresponds to placing two marks in separate squares. It is like we are playing two classical tic tac toe at same time.

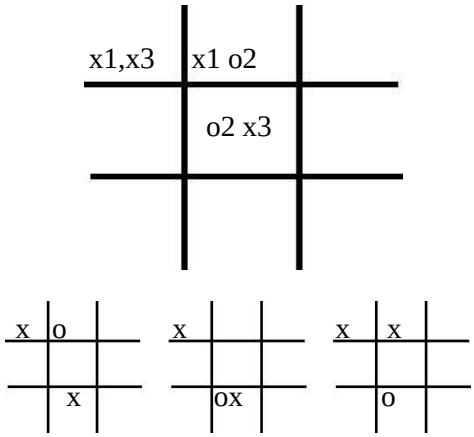
ENTANGLEMENT:-





A situation arises when x and o both can be in square 2. On measurement, if x collapses in square 2 then o will be in square 5. And if o is in square 2 then x will be in square 1. That is they share entanglement: measurement outcome of one influences others.

CYCLIC ENTANGLEMENT:-



X1 is entangled with o2 because they share same square, o2 is entangled with x3 in the central square and x1 is entangled with x3 in the corner square.

If x1 collapses to square 2, o2 collapses into square 5 and x3 into square 1 so x1 collapses to square 2. This is termed as cyclic entanglement.

WINNING:-

Player wins as soon as there are three classical marks in a row, quantum superposed marks are not counted.

FLIPIT: THE GAME [77]

It is an infinite game between a defender and an (advanced persistent threat) attacker for the control of a resource.

The game goes like: player 1 makes a move to gain control of the resource. It remains in this state till opponent takes over.

TERMINOLOGY:

Strategic form game is denoted by $< n, \{a_i\}, \{u_i\} >$. Set of players is given by $n = \{0, 1\}$, where 0 corresponds to defender, 1 corresponds to attacker.

Game starts at time t=0 and continues indefinitely. Game is defined as "control of resource r is altered between attacker and defender". We define a parameter $c_i(t) = 1$, if player i is controlling resource at time t, otherwise $c_i(t) = 0$. Initial state is $c_0(0) = 1$, the game starts with defender controlling the resource. Move is defined as player takes control of the resource by making a move/flip. If defender moves at time t, $c_0(t) = 1$. If attacker moves at time t', $c_0(t') = 0$. Total control time of player i until time t, $g_i(t) = \int_0^t c_i(t)\, dt$; $n_i(t)$ is player i's moves until time t; k_i constant cost for each move. Net benefit for player i, $b_i(t) = g_i(t) - n_i(t)\, k_i$. Utility is defined as the average benefit per unit time $u_i(t) = b_i(t) / t = (g_i(t) / t) - (n_i(t) / t)\, k_i = \gamma_i(t) - \alpha_i(t)\, k_i$, where $\gamma_i(t)$ is the average gain rate, $\alpha_i(t)$ is the average move rate of player i up to time t. As time $t \rightarrow \infty$, $\gamma_i(t) \rightarrow \gamma_i$ and $\alpha_i(t) \rightarrow \alpha_i$ utility $u_i = \lim t \rightarrow \infty\, u_i(t) = \gamma_i - \alpha_i\, k_i$. Before start, player rate $\alpha_i > 0$. As game progresses i moves at rate α_i after $\delta_i = 1/\alpha i$ units of time. Phase is given by $[0, \delta_i]$. Move rate is given by α_0 for the defender and α_1 for the attacker. Player know each other's flipping period not exact flipping time.

For periodic moving strategy

$A_0 = a_1 = p = \{p_\alpha \mid \alpha > 0\}$, k_0, k_1 are costs per unit move; 0, 1 denote the players. Flipit is a strategic form game $< n, \{a_i\}, \{u_i\} >$. Flipit(a_0, a_1). Periodic strategy: flipit (p, p), where $p = \{p_\alpha \mid \alpha > 0\}$. For every action profile $(p_{\alpha 0}, p_{\alpha 1}) \in p^2$

Player's utilities are

$U_i (p_{\alpha i}, p_{\alpha -i}) = 1 - (\alpha_{-i} / 2\, \alpha_i) - \alpha_i\, k_i$, if $\alpha_i \geq \alpha_{-i}\, u_i (p_{\alpha i}, p_{\alpha -i})$

$= (\alpha_i / 2\, \alpha_{-i}) - \alpha_i\, k_i$, otherwise.

ANALYSIS OF THE GAME OF FOOTBALL USING GAME THEORY [78]

This game is played between two teams. At any point of time, one plays offensive and other one defensive. Both have an equally effective running and passing game.

If team a plays offensive and the opponent, team b, *calls the play* i.e., opposing team defends against the move of team a - meaning that they would be defending against the run when team a would run, and would also be defending against the pass when team a would be passing- then team a can expect to lose five yards.

On the other hand, if the opponent calls the wrong play i.e., expects team a to run when team a passes, or expects team a to pass when it runs, team a would definitely get 5 yards on the play. Payoff matrix is given by

		Defense	
		Run	Pass
Offense	Run	-5,5	5,-5
	Pass	5,-5	-5,5

Winning strategy would be to play a mixed game, each move with probability = ½.

Now, imagine if team a has a better running game than passing, and when defence expects team a to pass but the team a runs, then tam a would gain 10 yards instead of 5.

Payoff matrix is given by

		Defense	
		Run	Pass
Offense	Run	-5,5	10,-10
	Pass	5,-5	-5,5

This way, running is a better strategy than passing. The winning strategy can be formulated as running with probability q and passing

With probability (1-q), while the opponent remains indifferent in between its two strategies.

Expected gain:

If defense defends against the	= 5q – 5(1-q)	= 10q - 5

Run		
If defense defends against the Pass	= -10 q + 5 (1-q)	= 5 – 15q

At equilibrium theses gains must be equal, so

$10q – 5 = 5 – 15q$

$Q= 2/5$

MOTIVATION

Applications of conventional game theory have played an important role in many modern strategic decision making processes including diplomacy, economics, national security, and business.

The method of recognizing formal game-theoretic solutions within domain-specific applications may also extend to quantum game-theoretic concepts.

In addition to decision making applications, game theory may also serve as a model for understanding competitive processes such as those found in ecological or social systems. It is an outstanding question to assess whether quantum game theory can provide new methods to these studies.

Several non-cooperative processes underlying existing approaches to the development of quantum technology including quantum control, quantum error correction, and fault-tolerant quantum operations. Each of these applications areas require a solution to the competition between the user and the environment, which may be considered to be 'player' in the game theoretic setting.

There are many situations for which distributed quantum resources offer a more natural application, e.g., quantum networking, and such formulations are at least realistic if not necessary.

The current availability of prototype general-purpose, quantum processors provides opportunities for the continued study of quantum game theory. This will include experimental studies of how users interacts with quantum games as well as the translation of quantum strategies into real-world settings.

There has been a wealth of traditional game studies over recent centuries, from historical, anthropological, archaeological– and more recently ethnological and cultural – perspectives.

The development of games has gone hand-in-hand with the development of human civilisation [82].

There is now a wealth of computational game studies; games have always been a driving factor behind artificial intelligence (ai) and machine learning (ml) research since the inception of these fields in the 1950s [83], and now game ai is maturing as a research field in its own right [84].

OBJECTIVES

The objectives to be pursued are initially proposed as under:

- To apply game theory to traditional Indian field games
  - A) To analyze classical game theory strategies,
  - B) To apply quantum game theory to find winning strategies
- To apply game theory to traditional Indian board games
  - A) To analyze classical game theory strategies,
  - B) To apply quantum game theory to find winning strategies
- To optimize and develop better suitable algorithm based on better winning strategies
- To undertake detailed comparative analysis of classical and quantum strategies based on the following parameters :
  - A) Efficiency
  - B) Time
  - C) Resources
- To explore the applications of quantum game theory to some real world problems.

METHODOLOGY:

It is proposed to apply game theory to traditional Indian field games to analyze classical game theory strategies and to apply quantum game theory to find winning strategies. For the same, the initial effort would be to try to application of game theory to some classical Indian field games, by studying examples where game theory has been applied to field games. Example of the game of football is given in the literature review. Then, the endeavor would be to find out the winning strategies for these games. And later on, it is intended to figure out if quantum game theory can be applied to find better winning strategies.

Similiarly, it is proposed to apply game theory to traditional Indian board games to analyze classical game theory strategies and to apply quantum game theory to find winning strategies. It is proposed to take quantum tic tac toe, covered in the literature review, as a standard example and intend to figure out the methodology for quantization of some Indian board games.

Then, it is proposed to undertake detailed comparative analysis of classical and quantum strategies based on the following parameters: efficiency, time and resources.

The effort would also be to generalize the classical and quantum strategies to find out generalized strategies, to try optimization and development of better suitable algorithm based on these winning strategies, and to explore the application of quantum game theory to some real world problems.

We also propose to explore the applications of quantum game theory to some real world problems. It is proposed to adopt a methodology similar to the kind exemplified in the following analysis.

GENERALIZED QUANTIZATION SCHEME FOR NON-ZERO SUM GAMES [71]

Battle of the sexes is an interesting static game of complete information. In its usual exposition two players, Alice and bob, try to decide somewhere to spend saturday evening. Alice wants to go for opera while bob is interested in watching TV at home and both would prefer to spend the evening together. The game is represented by the following payoff matrix:

	Bob		
Alice		O	T
	O	(α, β)	(σ, σ)
	T	(σ, σ)	(β, α)

Where O and T represent opera and TV, respectively, and α, β, σ are the payoffs for players for different choices of strategies, such that, $\alpha > \beta > \sigma$. There exist two Nash Equilibria, (O, O) and (T , T), in the classical form of the game. In the absence of any communication between Alice and Bob, there is a dilemma as the Nash Equilibria (O, O) suit Alice whereas bob prefers the Nash Equilibria (T, T). As a result both players could end up with worst payoff in the case where they play mismatched strategies.

Here we present a generalized quantization scheme by redefining unitary operators in the Marinatto and weber scheme: let Alice and Bob be given the following initial state:

$|\psi_{in}\rangle = \cos \gamma/2 \, |OO\rangle + i \sin \gamma /2 \, |TT\rangle$.

Where $|O\rangle$ and $|T\rangle$ represent the vectors in the strategy space corresponding to opera and TV, Respectively and $\gamma \in [0, \pi/2]$. The strategies of the two players are represented by the unitary Operator, u_i, which is of the form

$U_i = \cos \Theta_i/2 \, R_i + \sin \Theta_i/2 \, C_i$,

Where i = 1 or 2 and R, C are the unitary operators defined as

$R|O\rangle = e^{i\varphi i} |O\rangle$, $R|T\rangle = e^{-i\varphi i} |T\rangle$,

$C|O\rangle = -|T\rangle$, $C|T\rangle = |O\rangle$.

Following [4], we restrict our treatment to a two-parameter set of strategies for mathematical simplicity. After the application of the strategies, the initial state (1) transforms into

$|\psi_f\rangle = (u_1 \otimes u_2) \, | \psi_{in} \rangle$

And using equations (2) and (3) the above expression becomes

$|\psi_f\rangle = \cos \gamma /2 \, [\cos \Theta_1/2 \, \cos \Theta_2/2 \, e^{i(\varphi 1 + \varphi 2)} |OO\rangle - \cos \Theta_1/2 \, \sin \Theta_2/2 \, e^{i\varphi 1} |OT\rangle - \cos \Theta_2/2 \, \sin \Theta_1/2 \, e^{i\varphi 2} |to\rangle + \sin \Theta_1/2 \, \sin \Theta_2/2 \, |TT\rangle] + i \sin \gamma /2 [\cos \Theta_1/2 \, \cos \Theta_2/2 \, e^{-i(\varphi 1 + \varphi 2)} |TT\rangle + \cos \Theta_1/2 \, \sin \Theta_2/2 \, e^{-i\varphi 1} |TO\rangle + \cos \Theta_2/2 \, \sin \Theta_1/2 \, e^{-i\varphi 2} |OT\rangle + \sin \Theta_1/2 \, \sin \Theta_2/2 \, |OO\rangle]$

The payoff operators for Alice and Bob are

$P_A = \alpha \, p_{OO} + \beta \, p_{TT} + \sigma \, (p_{OT} + p_{TO})$ $\qquad$ $p_B = \alpha \, p_{TT} + \beta \, p_{OO} + \sigma \, (p_{OT} + p_{TO})$

Where

$P_{OO} = |\psi_{OO}\rangle\langle\psi_{OO}|$, $|\psi_{OO}\rangle = \cos \delta/2 |OO\rangle + i \sin \delta/2 |TT\rangle$,

$P_{TT} = |\psi_{TT}\rangle\langle\psi_{TT}|$, $|\psi_{TT}\rangle = \cos \delta/2 |TT\rangle + i \sin \delta/2 |OO\rangle$,

$P_{TO} = |\psi_{TO}\rangle\langle\psi_{TO}|$, $|\psi_{TO}\rangle = \cos \delta/2 |TO\rangle - i \sin \delta/2 |OT\rangle$,

$P_{OT} = |\psi_{OT}\rangle\langle\psi_{OT}|$, $|\psi_{OT}\rangle = \cos \delta/2 |OT\rangle - i \sin \delta/2 |TO\rangle$,

With $\delta \in [0, \pi/2]$.

The above payoff operators reduce to that of eisert's scheme for δ equal to γ, which represents the entanglement of the initial state. And for δ = 0 above operators transform into that of marinatto and weber's

Scheme. In generalized quantization scheme payoff for the players is calculated as

$\$_a(\theta_1, \varphi_1, \theta_2, \varphi_2) = tr(p_a\rho_f)$, $\$_b(\theta_1, \varphi_1, \theta_2, \varphi_2) = tr(p_b\rho_f)$,

Where $\rho_f = |\psi_f\rangle\langle\psi_f|$ is the density matrix for the quantum state given by (5) and tr represents

The trace of a matrix. Using equations (5), (6), (8) the payoffs for players are obtained as

$\$_a(\theta_1, \varphi_1, \theta_2, \varphi_2) = \cos^2 \theta_1/2 \cos^2 \theta_2/2 [\eta \sin^2 \gamma/2 + \xi \cos^2 \gamma/2 + \chi \cos 2(\varphi_1 + \varphi_2) \sin \gamma - \sigma] + \sin^2 \theta_1/2$

$\sin^2 \theta_2/2(\eta \cos^2 \gamma/2 + \xi \sin^2 \gamma/2 - \chi \sin \gamma - \sigma) + [(\alpha + \beta - 2\sigma) \sin \gamma - 2\chi]/4 \sin \theta_1 \sin \theta_2 \sin(\varphi_1 + \varphi_2) + \sigma$

$\$_b(\theta_1, \varphi_1, \theta_2, \varphi_2) = \cos^2 \theta_1/2 \cos^2 \theta_2/2 [\xi \sin^2 \gamma/2 + \eta \cos^2 \gamma/2 - \chi \cos 2(\varphi_1 + \varphi_2) \sin \gamma - \sigma] + \sin^2 \theta_1/2$

$\sin^2 \theta_2/2(\xi \cos^2 \gamma/2 + \eta \sin^2 \gamma/2 + \chi \sin \gamma - \sigma) + [(\alpha + \beta - 2\sigma) \sin \gamma - 2\chi]/4 \sin \theta_1 \sin \theta_2 \sin(\varphi_1 + \varphi_2) + \sigma$

Where $\xi = \alpha \cos^2 \delta/2 + \beta \sin^2 \delta/2$, $\eta = \alpha \sin^2 \delta/2 + \beta \cos^2 \delta/2$ and $\chi = (\alpha - \beta)/2 \sin \delta$. Classical results can easily be found from equations (9a), (9b) by simply unentangling the initial quantum state of the game, i.e., letting γ = 0. Furthermore, all the results of both Marinatto and weber [72] and Eisert et al [73] are embedded in these payoffs. For different combinations of δ and φ's there exist the following possibilities:

Case (a). When $\delta = 0$ and

(I) $\Phi_1 = 0, \varphi_2 = 0$, then the payoffs for the players from equations (9*a*), (9*b*) reduce to

$\$_a(\theta_1, \varphi_1, \theta_2, \varphi_2) = \cos^2 \theta_1/2 [\cos^2 \theta_2/2 (\alpha + \beta - 2\sigma) - \alpha \sin^2 \gamma/2 - \beta \cos^2 \gamma/2 + \sigma] + \cos^2 \theta_2/2 (-\alpha \sin^2 \gamma/2 - \beta \cos^2 \gamma/2 + \sigma) + \alpha \sin^2 \gamma/2 + \beta \cos^2 \gamma/2$

(10*a*)

$\$_b(\theta_1, \varphi_1, \theta_2, \varphi_2) = \cos^2 \theta_2/2 [\cos^2 \theta_1/2 (\alpha + \beta - 2\sigma) - \beta \sin^2 \gamma/2 - \alpha \cos^2 \gamma/2 + \sigma] + \cos^2 \theta_1/2 (-\beta \sin^2 \gamma/2 - \alpha \cos^2 \gamma/2 + \sigma) + \beta \sin^2 \gamma/2 + \alpha \cos^2 \gamma/2.$ (10*b*)

These payoffs are the same as found by marinatto and weber [5] when the players are applying the identity operators i_1 and i_2 with probabilities $\cos^2 \theta_1/2$ and $\cos^2 \theta_2/2$, respectively, for the given initial quantum state of the form (1).

(II) $\Phi_1 + \varphi_2 = \pi/2$, then equations (9a), (9b) reduce to

$\$_a(\theta_1, \varphi_1, \theta_2, \varphi_2) = \cos^2 \theta_1/2 [\cos^2 \theta_2/2 (\alpha + \beta - 2\sigma) - \alpha \sin^2 \gamma/2 - \beta \cos^2 \gamma/2 + \sigma] + \cos^2 \theta_2/2 (-\alpha \sin^2 \gamma/2 - \beta \cos^2 \gamma/2 + \sigma) + \alpha \sin^2 \gamma/2 + \beta \cos^2 \gamma/2 + (\alpha + \beta - 2\sigma)/4 \sin \gamma \sin \theta_1 \sin \theta_2,$ (11*a*)

$\$_b(\theta_1, \varphi_1, \theta_2, \varphi_2) = \cos^2 \theta_2/2 [\cos^2 \theta_1/2 (\alpha + \beta - 2\sigma) - \beta \sin^2 \gamma/2 - \alpha \cos^2 \gamma/2 + \sigma] + \cos^2 \theta_1/2 (-\beta \sin^2 \gamma/ 2 - \alpha \cos^2 \gamma/2 + \sigma) + \beta \sin^2 \gamma/2 + \alpha \cos^2 \gamma/2 + (\alpha + \beta - 2\sigma)/4 \sin \gamma \sin \theta_1 \sin \theta_2.$ (11*b*)

In the context of Marinatto and Weber scheme [72, 74] above payoffs for the two players Correspond to a situation when the strategies of the players are linear combination of operators i And flip operator σ_x of the form $o_i = \sqrt{p_i}\, i + \sqrt{1 - p_i}\, \sigma_x$ with $p_i = \cos^2 \theta_i/2$ and initial entangled state is of the form given in equation (1).

Case (b). When $\delta = \gamma$ and

$\Phi_1 \neq 0, \varphi_2 \neq 0$ then payoffs given by the equations (9*a*), (9*b*), very interestingly, change to the payoffs as if the game has been quantized using Eisert *et al* [73] scheme for the initial quantum state of the form (1). In this situation the payoffs for both the players are $\$_a(\theta_1, \varphi_1, \theta_2, \varphi_2) = \cos^2 \theta_1/2 \cos^2 \theta_2/2 [\eta_1 \sin^2 \gamma/2 + \xi_1 \cos^2 \gamma/2 + \chi_1 \cos 2(\varphi_1 + \varphi_2) - \sigma] + \sin^2 \theta_1/2$

$\sin^2 \theta_2/2(\eta_1 \cos^2 \gamma/2 + \xi_1 \sin^2 \gamma/2 - \chi_1 - \sigma) + (\beta - \sigma)/2 \sin \gamma \sin \theta_1 \sin \theta_2 \sin(\varphi_1 + \varphi_2) + \sigma$

$\$_b(\theta_1, \varphi_1, \theta_2, \varphi_2) = \cos^2 \theta_1/2 \cos^2 \theta_2/2 [\xi_1 \sin^2 \gamma/2 + \eta_1 \cos^2 \gamma/2 - \chi_1 \cos 2(\varphi_1 + \varphi_2) - \sigma] + \sin^2 \theta_1/2$

$\sin^2 \theta_2/2(\xi_1 \cos^2 \gamma/2 + \eta_1 \sin^2 \gamma/2 + \chi_1 - \sigma) + (\alpha - \sigma) \sin \gamma \sin \theta_1 \sin \theta_2 \sin(\varphi_1 + \varphi_2) + \sigma$

Where $\xi 1 = \alpha \cos 2 \gamma/2 + \beta \sin 2 \gamma/2$, $\eta 1 = \alpha \sin 2 \gamma/2 + \beta \cos 2 \gamma/2$ and $\chi 1 = (\alpha - \beta)/2 \sin^2 \gamma$. To draw a better comparison we take $\delta = \gamma = \pi/2$ then the payoffs given by equations (12) reduce to

$\$_a(\theta_1, \varphi_1, \theta_2, \varphi_2) = (\alpha - \sigma) \cos^2 \theta_1/2 \cos^2 \theta_2/2 \sin^2 (\varphi_1 + \varphi_2) + (\beta - \sigma) [\cos \theta_1/2 \cos \theta_2/2 \sin(\varphi_1 + \varphi_2) + \sin \theta_1/2 \sin \theta_2/2]^2 + \sigma$ (13*a*)

$\$_b(\theta_1, \varphi_1, \theta_2, \varphi_2) = (\alpha - \sigma)\,[\cos\theta_1/2 \cos\theta_2/2 \sin(\varphi_1 + \varphi_2) + \sin\theta_1/2 \sin\theta_2/2]^2 + (\beta - \sigma) \cos^2\theta_1/2 \cos^2\theta_2/2 \sin^2(\varphi_1 + \varphi_2) + \sigma.$ (13*b*)

The payoffs given in equations (13) have already been found by du *et al* [75] through the eisert *et al* scheme [73].

(ii) $\varphi_1 = \varphi_2 = 0$ then, as shown by eisert *et al* [73, 76], one gets classical payoffs with mixed strategies when one parameter set of strategies is used for the quantization of the game. For a better comparison putting $\gamma = \delta = \pi 2$

And $\varphi_1 = \varphi_2 = 0$ in equations (12*a*) and (12*b*) the same situation occurs and the payoffs reduce to

$\$_a(\theta_1, \varphi_1, \theta_2, \varphi_2) = \alpha \cos^2\theta_1/2 \cos^2\theta_2/2 + \beta \sin^2\theta_1/2 \sin^2\theta_2/2 + \sigma\,(\cos^2\theta_1/2 \sin^2\theta_2/2 + \sin^2\theta_1/2 \cos^2\theta_2/2$ (14*a*)

$\$_b(\theta_1, \varphi_1, \theta_2, \varphi_2) = \beta \cos^2\theta_1/2 \cos^2\theta_2/2 + \alpha \sin^2\theta_1/2 \sin^2\theta_2/2 + \sigma\,(\cos^2\theta_1/2 \sin^2\theta_2/2 + \sin^2\theta_1/2 \cos^2\theta_2/2$ (14*b*)

In this case the game behaves just like classical game where the players are playing mixed strategies with probabilities $\cos^2\theta_1/2$ and $\cos^2\theta_2/2$ respectively.

Case (c). When $\delta \neq \gamma$ and $\varphi_1 = 0$, $\varphi_2 = 0$, then payoffs given by the equations (9*a*), (9*b*) reduce to

$\$_a(\theta_1, \varphi_1, \theta_2, \varphi_2) = \cos^2\theta_1/2\,[\cos^2\theta_2/2\,(\alpha + \beta - 2\sigma) - \alpha \sin^2(\gamma - \delta)/2 - \beta \cos^2(\gamma - \delta)/2 + \sigma\,]$

$+ \cos^2\theta_2/2\,[-\alpha \sin 2(\gamma - \delta)/2 - \beta \cos^2(\gamma - \delta)/2 + \sigma\,] + \alpha \sin^2(\gamma - \delta)/2 + \beta \cos^2(\gamma - \delta)/2$ (15*a*)

$\$_b(\theta_1, \varphi_1, \theta_2, \varphi_2) = \cos^2\theta_1/2\,[\cos^2\theta_2/2\,(\alpha + \beta - 2\sigma) - \beta \sin^2(\gamma - \delta)/2 - \alpha \cos^2(\gamma - \delta)/2 + \sigma\,] + \cos^2\theta_2/2$

$[-\beta \sin^2(\gamma - \delta)/2 - \alpha \cos^2(\gamma - \delta)/2 + \sigma\,] + \beta \sin^2(\gamma - \delta)/2 + \alpha \cos^2(\gamma - \delta)/2$ (15*b*)

These payoffs are equivalent to that of marinatto and weber [72] when γ replaced with $\gamma - \delta$.

Case (d). When $\delta \neq 0$ and $\gamma = 0$ then from equations (9*a*), (9*b*) the payoffs of the players reduce to

$\$_a(\theta_1, \varphi_1, \varphi_2, \theta_2) = \cos^2\theta_1/2\,[\cos^2\theta_2/2\,(\alpha + \beta - 2\sigma) - \alpha \sin^2\delta/2 - \beta \cos^2\delta/2 + \sigma\,] + \cos^2\theta_2/2\,(-\alpha \sin^2\delta/2 - \beta \cos^2\delta/2 + \sigma) + \alpha \sin^2\delta/2 + \beta \cos^2\delta/2 - (\alpha - \beta)/2 \sin\delta \sin\theta_1 \sin\theta_2 \sin(\varphi_1 + \varphi_2)$ (16*a*)

$\$_b(\theta_1, \varphi_1, \varphi_2, \theta_2) = \cos^2\theta_2/2\,[\cos^2\theta_1/2\,(\alpha + \beta - 2\sigma) - \beta \sin^2\delta/2 - \alpha \cos^2\delta/2 + \sigma] + \cos^2\theta_1/2\,(-\beta \sin^2\delta/2 - \alpha \cos^2\delta/2 + \sigma) + \beta \sin^2\delta/2 + \alpha \cos^2\delta/2 + (\alpha - \beta)/2 \sin\delta \sin\theta_1 \sin\theta_2 \sin(\varphi_1 + \varphi_2).$ (16*b*)

(I) This shows that the measurement plays a crucial role in quantum games as even if the initial state is unentangled, i.e., $\gamma = 0$, arbiter can still apply entangled basis for the measurement to obtain quantum

Mechanical results. The above payoffs are similar to that of Marinatto and weber for the battle of the sexes games if δ is replaced by γ.

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